MATHEMATICS

LEAVING CERTIFICATE
HIGHER LEVEL

FACTOR THEOREM

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Algebra

1) Factor theorem questions

On the old maths syllabus the Examiner could ask you to prove a thing called "The Factor Theorem". The good news is that you cannot be asked to prove this anymore, however, the bad news is that you can be asked questions based on it. Below you will see 6 different types of factor theorem questions:

Type 1: Verifying and finding factors.

Example 1:
Verify that (x – 1) is a factor of $x^3 + 2x^2 - x - 2$ and find the other two factors.

NB: There are two ways we can prove (x – 1) is a factor.
Method 1: Subbing In

If $x - 1$ is a factor,
then $x = 1$ is a root

let's check if this is true by subbing in:

$= (1)^3 + 2(1)^2 - (1) - 2$
$= 1 + 2 - 1 - 2$
$= 0$

=> $x = 1$ is a root
=> $x - 1$ is a factor

But I haven't fully answered the question, since they also asked us to find the other two factors, therefore we need to use another method…

Method 2: Using Long Division (this method is more useful when asked to find the other factors/roots)

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \\ & \ominus x^3 \overset{\oplus}{-} x^2 \\ \hline & \quad 3x^2 - x - 2 \\ & \quad \ominus 3x^2 \overset{\oplus}{-} 3x \\ \hline & \qquad 2x - 2 \\ & \qquad \ominus 2x \overset{\oplus}{-} 2 \\ \hline & \qquad\quad 0 \end{array}$$

No remainder ∴ $x - 1$ is a factor

Now we can factorise what's on 'top of the roof' to find the other 2 factors:

$x^2 + 3x + 2$
$= (x + 2)(x + 1)$

Comments:

Question 1.1

If $P(x) = 6 + x - 4x^2 + x^3$, show that $(3 - x)$ is a factor of $P(x)$.
Find the other two factors of $P(x)$.

Comments:

Type 2: Solving.

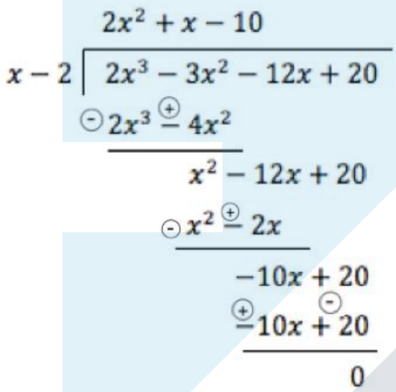
Example 2:
Solve the equation: $\mathbf{2x^3 - 3x^2 - 12x + 20 = 0}$

Type 2 is very similar to Type 1. We just need to do one extra step at the start, by finding a factor by trial and error.
Tip: *Find the factors of the constant and use these as possible values for x when subbing in*.

Try $x = 1$: $= 2(1)^3 - 3(1)^2 - 12(1) + 20$
$= 7$
$\Rightarrow x = 1$ is not a root

Try $x = -1$: $= 2(-1)^3 - 3(-1)^2 - 12(-1) + 20$
$= 27$
$\Rightarrow x = -1$ is not a root

Try $x = 2$: $= 2(2)^3 - 3(2)^2 - 12(2) + 20$
$= 0$
$\Rightarrow x = 2$ is a root
$\Rightarrow x - 2$ is a factor



Now we can factorise what's on 'top of the roof' to find the other 2 factors:

$2x^2 + x - 10$
$= (2x + 5)(x - 2)$

From here we can find the roots:

$\Rightarrow 2x + 5 = 0$	$x - 2 = 0$
$2x = -5$	$x = 2$
$x = \frac{-5}{2}$	

Comments:

Question 1.2

Given that one of the roots is an integer, solve the equation:

$6x^3 - 29x^2 + 36x - 9 = 0$

Comments:

Type 3: Finding an unknown value.

Example 3:

If $(x + 1)$ is a factor of $x^3 + 5x^2 + kx - 12$, find the value of k and hence find the other two factors of the cubic expression.

Step 1: Sub root into function and solve to find unknown value

Step 2: (If asked) use long division to find other factors

If $x + 1$ is a factor, then $x = -1$ is a root. So when we sub $x = -1$ in we must get 0:

$(-1)^3 + 5(-1)^2 + k(-1) - 12 = 0$

$-k - 8 = 0$

$-8 = k$

Now to find the other factors we continue just like 'Type 1':

$$x^2 + 4x - 12$$

$$x + 1 \overline{\smash{)}\, x^3 + 5x^2 - 8x - 12}$$

$$\ominus \; x^3 + x^2$$

$$4x^2 - 8x - 12$$

$$\ominus \; 4x^2 + 4x$$

$$-12x - 12$$

$$\oplus \; -12x - 12$$

$$0$$

Now we can factorise what's on 'top of the roof' to find the other 2 factors:

$x^2 + 4x - 12$

$= (x + 6)(x - 2)$

Comments:

Question 1.3

If $(2x - 1)$ is a factor of the polynomial $P(x) = 2x^3 - 5x^2 - kx + 3$, find the value of k.
Find the other two factors of $P(x)$.

Comments:

Type 4: Finding unknown values when given 2 factors.

Example 4:
Let $f(x) = 2x^3 - ax^2 - bx + 42$, where a and b are constants. Given that $(x - 2)$ and $(x + 3)$ are factors of $f(x)$, find the values of a and b.

Step1: Sub the roots in.
Step 2: Solve the simultaneous equations.

If $x - 2$ is a factor $\Longrightarrow x = 2$ is a root.

$2(2)^3 - a(2)^2 - b(2) + 42 = 0$
$-4a - 2b + 58 = 0$ ①

If $x + 3$ is a factor $\Longrightarrow x = -3$ is a root.

$2(-3)^3 - a(-3)^2 - b(-3) + 42 = 0$
$-9a + 3b - 12 = 0$ ②

Now using simultaneous equations:

3 X ① $-12a - 6b + 174 = 0$

2 X ② $-18a + 6b - 24 = 0$

$-30a + 150 = 0$

$-30a = -150$

$$a = \frac{-150}{-30}$$

$a = 5$

Subbing back into ①:

$-4(5) - 2b + 58 = 0$
$-2b + 38 = 0$
$-2b = -38$
$b = \frac{-38}{-2}$
$b = 19$

Comments:

Question 1.4

If $(x-5)$ and $(x-3)$ are factors of $x^3-7x^2+ax+b=0$, then find the values of a and b.

Type 5: Finding unknown values when given 1 factor.

Example 5:
Given that $x^2 - ax - 3$ is a factor of
$x^3 - 5x^2 + bx + 9$ where $a, b \in R$, find the value of a and the value of b.

Step 1. Use long division

Step 2. Calculate the letters by putting the remainder = 0

$$x + (-5+a)$$

$$x^2 - ax - 3 \,\overline{)\, x^3 - 5x^2 + bx + 9}$$

$$\ominus\; x^3 \oplus ax^2 \oplus 3x$$

$$(-5+a)x^2 + (b+3)x + 9$$

$$\ominus\; (-5+a)x^2 \ominus (5a-a^2)x \ominus (15-3a)$$

$$(b+3-5a+a^2)x + (-6+3a)$$

Now, this remainder should be 0, so we can put the x coefficient equal to 0

and also put the constant equal to 0:

$b+3-5a+a^2=0$	$-6+3a=0$
$b+3-5(2)+(2)^2=0$	$3a=6$
$b-3=0$	$a=\frac{6}{3}$
$b=3$	$a=2$

Comments:

Question 1.5

Given that $x^2 - 2x - 3$ is a factor of $sx^3 + 8x^2 + rx + 6$, find the values of s and r.

Comments:

Type 6: Proving "letters equal letters"

Example 6:

If $x^2 - ax + b$ is a factor of $x^3 + 2ax^2 + 4bx + c$ show that

(i) $b = -a^2$

(ii) $c = -3a^3$

Step 1: Long Division

Step 2: Calculate the letters by putting the remainder = 0

Step 3: Manipulate what you have to give what the Examiner wants.

$$\begin{array}{r|l} & \boldsymbol{x + 3a} \\ \boldsymbol{x^2 - ax + b} & \boldsymbol{x^3 + 2ax^2 + 4bx + c} \\ & \ominus \boldsymbol{x^3} \oplus\!\!\!- \boldsymbol{ax^2} \ominus\!\!\!+ \boldsymbol{bx} \\ \hline & \boldsymbol{3ax^2 + 3bx + c} \\ & \ominus \boldsymbol{3ax^2} \oplus\!\!\!- \boldsymbol{3a^2x} \ominus\!\!\!+ \boldsymbol{3ab} \\ \hline & \boldsymbol{(3b + 3a^2)x + (c - 3ab)} \end{array}$$

Now, this remainder should be 0, so we can put the x coefficient equal to 0 and also put the constant equal to 0:

$3b + 3a^2 = 0$

$3b = -3a^2$

$b = -a^2$

$c = 3ab = 0$

$c = -3a(-a^2) = 0$

$c + 3a^3 = 0$

$c = -3a^3$

Comments:

Question 1.6

If $x^2 - px + q$ is a factor of $x^3 + 3px^2 + 3qx + r$

Prove that: (i) $q = -2p^2$

(ii) $r = -8p^3$

Comments:

2) Past and probable exam questions

Type of Question	Skill Needed	Hint	Year
Factor Theorem/Cubics			
$2x^3 + 5x^2 - 4x - 3$,	All factor theorem questions involve similar skills that will transfer into other chapters also. - Understanding a root satisfies an equation - Understanding a root is where the function crosses the X axis - Being able to find a root by trial and error (must be factors of the final number) - Being able to turn a root into a factor (go backwards!) - Long division or comparison method - Solving a quadratic	Root Factor Verify Solve	2018 2017 2015 2013
Simultaneous Equations			(P1)*
$2x + 3y - z = -4$ $3x + 2y + 2z = 14$ $x - 3z = -13$ $x^2 + xy + 2y^2 = 4$ $2x + 3y = -1.$	a) two equations two unknowns - set up equations using multiplication so that one of the letters cancel, solve and sub back in to find the other letter. b) three equations three unknowns - pick two equations and get rid of a letter. Pick a different combination of the equations and get rid of <u>the same letter.</u> - Solve the two new equations like a) above. - Sub answer back in to find the value of the two other letters. c) two equations one which contains a letter squared (x^2/y^2) - go to the equation which doesn't have a squared in it (linear equation). Isolate one of the letters in this equation. (Put by itself)	Solve Intersection Force answers	2018 2016 2013 2012 2010 *feature heavily on P2 and as tools in other topics.

	- substitute the isolated letter into the equation with a squared so now it only has one unknown. - Simplify and solve the quadratic before subbing both answers back in to the linear equation. Note 1: Sometimes the examiner will include fractions in simultaneous equation questions. Get rid of these separetly before beginning the question. Note 2: The examiner may give you a second question that says 'hence solve' where the simultaneous equation in this one looks extremely difficult however once you recognise it is the same as the first equation you can solve the question easily. Put your answers for part i) = to the thing taking the place of x in part ii) and solve. Note 3: In b) sometimes the examiner will have done some of the work for you where one of your equations given will only have two unknowns. Work on the other two to get rid of the letter not in the third equation.		
Inequalities			
$\frac{2x-3}{x+2} \geq 3$	- Get rid of fractions by multiplying both sides by the bottom squared when you are unsure if it is positive or negative. - Solve quadratic equation to find critical values - Have a method to use critical values to come up with your inequality solution. (Cheat) Note: If you multiply both sides by a positive whole number the crocodile stays the same. If you multiply by a negative it switches sides. Simply taking numbers across it acts like an equal sign in that you change a positive value to a negative and vice versa.	Do not simplify the question (LPC) X $\neq$	2018 2016 2013 2012
Miscellaneous			
	- Multiplying brackets - Getting rid of fractions	Solve	2019 2017 2015 2013

	- Dealing with surds - Completing the square - Nature of the roots (b_2-4ac) Note: completing the square may also come with looking for a max/min point and solving the quadratic. To get the max/max point change the sign of the number inside the bracket while keeping the number outside exactly as is.	$a(x+h)^2+k,$ 'has real roots'	2011
Logs and Indices			
$Q = e^{-\frac{0.693t}{5730}}$ $\log_2(x+2)+\log_2(x-2)=5.$	a) Log function* - usually a long question where you have to sub in values given in the question. - You will have to solve for a power (usually t) using logs. - Sometimes you will not have the full function to begin with and may have to find a constant. The examiner will give you two time periods in this case or state something in the question where at the beginning t = 0 - May sub values into the function to form a graph. b) Log Rules - use main rules of logs to write something in the form of another - log equations havent appeared yet but use the rules and then form a junior cert quadratic (the thing on the left to the power of the thing in the right = the thing	Solve Problem with powers – use logs	2017 2016 2014 2013 2012

$2^x + 2^{1-x} = 3.$	in the middle!) c) Indices - use the rules of indices to rewrite usually a quadratic, substitute in for the base, solve the quadratic and then solve for the original letter.		

Past Exam Questions

Exam Paper 2021, Question 2: **30m**

(b) $(x + 4)$ is a factor of $f(x) = x^3 + qx^2 - 22x + 56$, where $x \in \mathbb{R}$ and $q \in \mathbb{Z}$.

Show that $q = -5$, **and** find the three roots of $f(x)$.

Show:

Roots = (,   ,   )

Exam Paper 2017, Question 5: 25m

The function f is such that $f(x) = 2x^3 + 5x^2 - 4x - 3$, where $x \in \mathbb{R}$.

(a) Show that $x = -3$ is a root of $f(x)$ **and** find the other two roots.

<u>Exam Paper 2015, Question 2:</u> 25m

Solve the equation $x^3 - 3x^2 - 9x + 11 = 0$.
Write any irrational solution in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$.